

The incident wave in Aharonov-Bohm scattering wavefunction

^aSahng-kyoon.Yoo and ^bD.K.Park

^a*Department of Physics, Seonam University, Namwon 590-711, Korea*

^b*Department of Physics, Kyungnam University, Masan 631-701, Korea*

(February 1, 2008)

Abstract

It is shown that only the infinite angular momentum quantum states contribute to the incident wave in Aharonov-Bohm (AB) scattering. This result is clearly shown by recalculating the AB calculation with arbitrary decomposition of summation over the angular momentum quantum numbers in wave function. It is motivated from the fact that the pole contribution in the integral representation used by Jackiw is given by only the infinite angular momentum states, in which the closed contour integration involving this pole gives just the incident wave.

PACS No.:03.65.Bz, 03.65.Nk

Since Aharonov-Bohm(AB) [1] discovered that particles in the field free region exhibit the quantum mechanical interference, known as the AB effect, many physicists [2–10] have studied in detail the scattering mechanism until now. There are, however, some disagreements in the scattering wave function, in particular, in incident wave.

Some authors [2,3] advocate the plane wave as an incident wave, while others [1,5–9] derived the plane wave modulated by flux. This debate is still being in progress. In the former case, the commutativity in the two performances, i.e., summation over the angular momentum states and the asymptotic limit of Bessel function, in exact wavefunction must be tested, as pointed out by Hagen [4]. Meanwhile, for the latter case, the plane wave modulated by flux as an incident wave is derived by various method. The incident wave and the scattering amplitude were calculated by AB [1] for the first time, by decomposing the wavefunction into positive, negative and zero angular momentum states. (They have restricted the ratio of flux to flux quantum from -1 to 1). They deduced the incident wave exactly from the wavefunction, while the scattered wave for the non-forward direction in the asymptotic limit. Jackiw [5] found another method for the derivation of incident and scattered waves by using the integral representation of Bessel function. He obtained the incident wave from the contour enclosing the pole, and the scattered wave from the remaining straight line contour. The two methods mentioned above give the exactly same results.

In this paper, focusing on the latter case (modulated plane wave as an incident wave), we investigate how the summation over angular momentum states are related to the closed contour yielding the incident wave. We find that only the infinite angular momentum states correspond to the pole in wavefunction, while all the other finite states represent the scattered wave. This fact leads us to recalculate the AB wavefunction by dividing the summation not into positive, negative and zero component, as was done by AB, but with the introduction of general decomposition, as will be described below. We find that the deduced incident and scattered waves are same regardless of the way of decomposition. This means that in AB scattering the information on the incident wave cannot be involved in the finite angular

momentum quantum states. It can be interpreted classically that only the waves infinitely far away from the flux tube at origin do not scatter at all, while the remaining ones are deflected due to the AB potential. However, it is uncertain whether such a property is general one in all kinds of the long-ranged potentials such as Coulomb potential as well as AB potential. It is being in progress.

Let us begin with the Schrödinger equation representing a system affected by AB potential in 2D polar coordinates:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \phi} + i\alpha \right)^2 + k^2 \right] \psi = 0, \quad (1)$$

where $\alpha = -e\Phi/2\pi$, e is the particle charge, Φ is the magnetic flux, and k is the wave number. We choose the gauge in which $A_r = 0$ and $A_\phi = \Phi/2\pi r$.

The regular solution of this equation is given by

$$\psi = \sum_{m=-\infty}^{m=\infty} (-i)^{|m+\alpha|} J_{|m+\alpha|}(r') e^{im\phi}, \quad (2)$$

where $J_\nu(x)$ is the ordinary Bessel function of ν -th order and $r' = kr$. If one takes the flux

$$\alpha = N + \beta, \quad (3)$$

where N is an integer and $0 < \beta < 1$, then the summation can be decomposed into positive and negative parts, respectively:

$$\psi = \sum_{m=-N}^{\infty} (-i)^{m+\alpha} J_{m+\alpha}(r') e^{im\phi} + \sum_{m=-N-1}^{-\infty} (-i)^{-(m+\alpha)} J_{-(m+\alpha)}(r') e^{im\phi}. \quad (4)$$

The above summations over the angular momentum states can be performed by using the integral representation for the Bessel function [11] whose contour is given in figure 1:

$$J_\nu(x) = e^{i\nu\pi/2} \int_{Cs} \frac{dz}{2\pi} e^{-ix \cos z + i\nu z}. \quad (5)$$

Inserting Eq.(5) into Eq.(4) and changing the variable $z \rightarrow -z$ in the second term, we get

$$\psi = \left(\int_{Cs} - \int_{C-s} \right) \frac{dz}{2\pi} e^{-ir' \cos z - i\alpha z} \sum_{m=-N}^{\infty} e^{im(\phi-z)} \quad (6)$$

$$= \int_C \frac{dz}{2\pi} e^{ir' \cos(\pi-z)} e^{i\alpha(\phi-\pi)} \frac{e^{-i\beta z}}{1 - e^{-iz}}, \quad (7)$$

where the contour C_{-s} is the mirror of C_s , and the contour C is given in figure 2. Note that both the upper and lower contour in complex z -plane are shifted upward and downward by small positive quantity ϵ . This is to avoid the poles in real axis raised in the course of summation of Eq.(6). The contour C can be divided into two parts: closed contour involving poles in real axis (C_1) and straight line (C_2), which is drawn in figure 3. Jackiw obtained the incident wave from C_1 integration, while the scattered amplitude from C_2 integration.

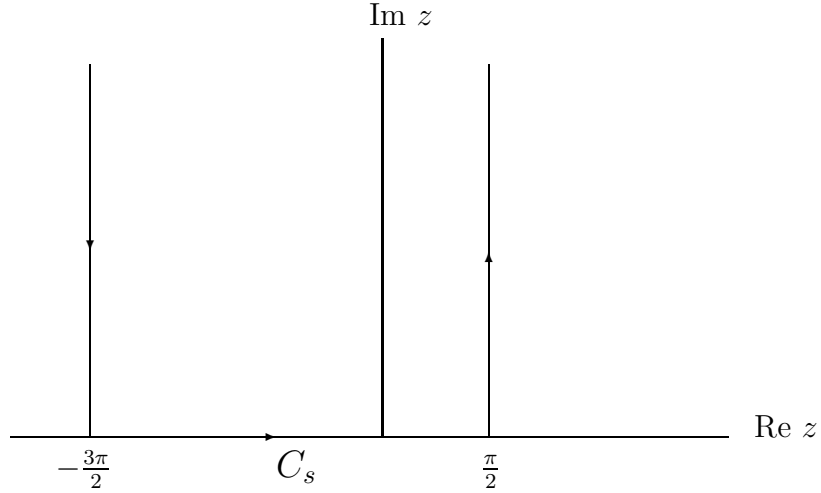


figure 1. Contour representing $J_\nu(x)$.

Here, let us take only finite summation in Eq.(6) to explore the relation between the incident wave and the partial wave. In case of the first term, assuming that M is large but finite, we have

$$\sum_{-N-1}^M e^{im(\phi-z)} = e^{-i(N+1)(\phi-z)} \frac{[1 - e^{i(M+N+2)(\phi-z)}]}{1 - e^{i(\phi-z)}}. \quad (8)$$

This is finite at $z = \phi$ as long as M is finite. This result is also true for the second term. Therefore, if we exclude the infinite contribution from the summation, then the poles in integrals do not appear. After all, only the infinite angular momentum states do contribute to the incident wave in AB scattering. It leads us to reconsider the AB calculation [1].

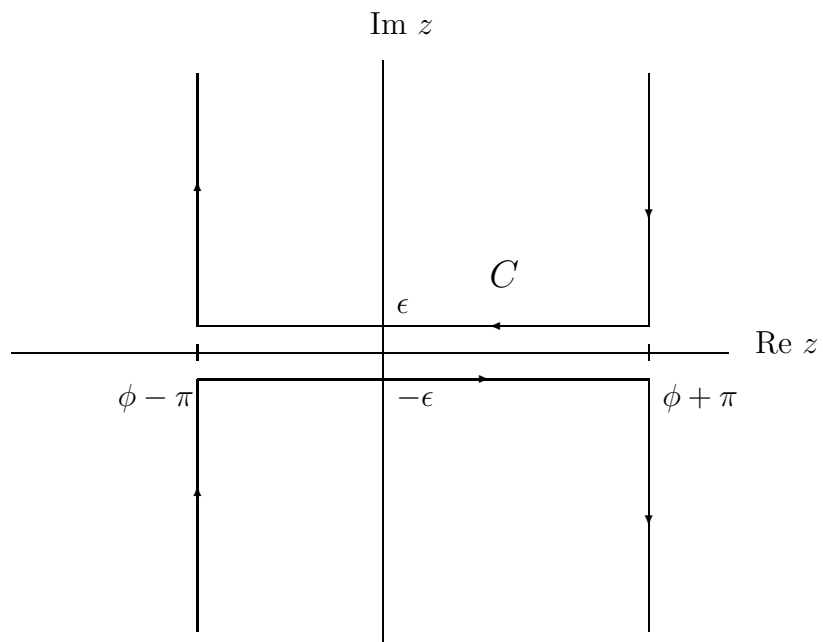


figure 2. Contour C representing the Eq.(7).

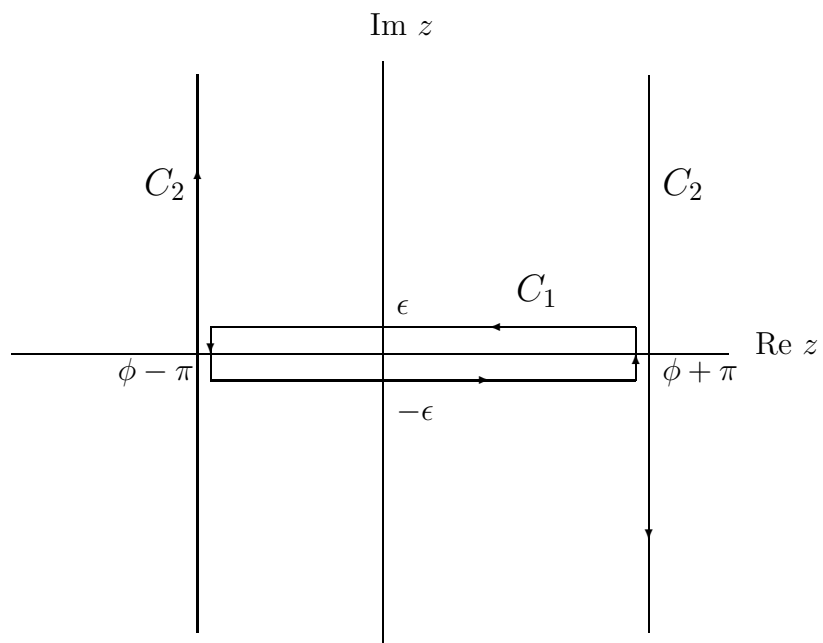


figure 3. Equivalent contour with figure 2.

Now, we generalize the AB calculation as follows:

The wavefunction of AB scattering is again given by Eq.(1). Unlike the AB calculation, we split ψ into three parts:

$$\psi_1 = \sum_{m=-N+M}^{\infty} (-i)^{(m+\alpha)} J_{(m+\alpha)}(r') e^{im\phi}, \quad (9)$$

$$\begin{aligned} \psi_2 &= \sum_{m=-N-1-M}^{-\infty} (-i)^{-(m+\alpha)} J_{-(m+\alpha)}(r') e^{im\phi} \\ &= \sum_{m=N+1+M}^{\infty} (-i)^{(m-\alpha)} J_{(m-\alpha)}(r') e^{-im\phi}, \end{aligned} \quad (10)$$

$$\psi_3 = \sum_{-N-M}^{-N+M-1} (-i)^{|m+\alpha|} J_{|m+\alpha|}(r') e^{im\phi}, \quad (11)$$

where M is a large positive integer. We then follow the procedure of AB. Note that ψ_2 is obtained from ψ_1 by replacing α by $-\alpha$ and ϕ by $-\phi$, and ψ_3 contains $2M$ terms unlike in AB. Let us consider first the cases of ψ_1 and ψ_2 . The differential equation satisfied by ψ_1 is

$$\frac{d\psi_1}{dr'} = -i \cos \phi \psi_1 + \frac{1}{2} (-i)^{M+\beta} e^{-i(N-M)\phi} \left[J_{M+\beta-1}(r') + i e^{-i\phi} J_{M+\beta}(r') \right], \quad (12)$$

whose solution is represented as an integral form:

$$\begin{aligned} \psi_1 &= A_1 \int_0^{r'} dr' e^{ir' \cos \phi} \left[J_{M+\beta-1}(r') + i e^{-i\phi} J_{M+\beta}(r') \right] \\ &\equiv A_1 [I_{11} - I_{12}], \end{aligned} \quad (13)$$

where

$$A_1 = \frac{1}{2} (-i)^{M+\beta} e^{-i(N-M)\phi} e^{-ir' \cos \phi}, \quad (14)$$

$$I_{11} = \int_0^{\infty} dr' e^{ir' \cos \phi} \left[J_{M+\beta-1}(r') + i e^{-i\phi} J_{M+\beta}(r') \right] \quad (15)$$

and

$$I_{12} = \int_{r'}^{\infty} dr' e^{ir' \cos \phi} \left[J_{M+\beta-1}(r') + i e^{-i\phi} J_{M+\beta}(r') \right]. \quad (16)$$

For ψ_2 , we take

$$\psi_2 \equiv A_2 [I_{21} - I_{22}]. \quad (17)$$

Here, A_2 , I_{21} and I_{22} are obtained by replacing $N \rightarrow -N - 1$, $\beta \rightarrow 1 - \beta$ and $\phi \rightarrow -\phi$ in A_1 , I_{11} and I_{12} , respectively. The first integrals in ψ_1 and ψ_2 , i.e., I_{11} and I_{21} can be calculated exactly:

$$\begin{aligned} A_1 I_{11} &= e^{-ir' \cos \phi - i\alpha \phi} \theta(\phi), \\ A_2 I_{21} &= e^{-ir' \cos \phi - i\alpha \phi} \theta(-\phi), \end{aligned} \quad (18)$$

where $\theta(x)$ is the usual step function. The sum of above terms gives exactly the same incident wave with that obtained by AB [1]. That is, this result is independent of finite M . Therefore, we can infer that only the infinite angular momentum quantum states yield the incident wave.

For I_{12} and I_{22} , we follow the similar procedure to AB with finite M , in order to obtain the scattering wave. Then, in the asymptotic limit, we obtain the followings:

$$\begin{aligned} A_1 I_{12} &= \frac{1}{2\sqrt{2\pi}} (-i)^{M+\beta} e^{-i(N-M)\phi} \\ &\times \frac{1}{\sqrt{r'}} \left\{ (-i)^{M+\beta-\frac{3}{2}} \frac{e^{ir'}}{1+\cos\phi} (1+e^{-i\phi}) + i^{M+\beta-\frac{3}{2}} \frac{e^{-ir'}}{1-\cos\phi} (1-e^{-i\phi}) \right\} \end{aligned} \quad (19)$$

and

$$A_2 I_{22} = A_1 I_{12} [N \rightarrow -N - 1, \beta \rightarrow 1 - \beta, \phi \rightarrow -\phi]. \quad (20)$$

Now we calculate the last term ψ_3 in the asymptotic limit:

$$\begin{aligned} \psi_3 &= \frac{1}{\sqrt{2\pi r'}} \left\{ e^{ir'} \frac{1}{1+e^{i\phi}} \left(1 - (-1)^M e^{iM\phi} \right) e^{-iN\phi} \left[(-i)^{\frac{1}{2}-2\beta-2M} e^{-iM\phi} + (-i)^{\frac{1}{2}+2\beta} \right] \right. \\ &\quad \left. + e^{-ir'} i^{\frac{1}{2}} \frac{e^{-i(N-M)\phi}}{1-e^{i\phi}} (1 - e^{2iM\phi}) \right\}. \end{aligned} \quad (21)$$

Finally, adding Eqs.(18), (19), (20) and (21), we obtain the incident and scattered wave which is independent of M :

$$\begin{aligned} \psi &= \psi_1 + \psi_2 + \psi_3 \\ &= e^{-ir' \cos \phi - i\alpha \phi} - \frac{i}{\sqrt{2\pi ir'}} \frac{e^{ir'}}{\cos \frac{\theta}{2}} e^{-i\frac{\phi}{2}} e^{-iN\phi} \sin \beta \pi. \end{aligned} \quad (22)$$

As we have shown above, the incident wave is obtained from $A_1 I_{11} + A_2 I_{21}$ and is independent of ψ_3 . We may choose M large enough but finite, so that ψ_3 is most of the contributions in summation and ψ_1 and ψ_2 contain only the infinite angular momenta. Since the incident wave are represented only by ψ_1 and ψ_2 , the information about the incident wave is involved in the $m = \pm\infty$. In classical mechanics, the large angular momentum in scattering corresponds to the large impact parameter from the scatterer located at the origin when velocity is constant. Since AB potential is long-ranged, it is natural that only the particles infinitely far away from the flux tube keep the information about the incident wave, and all the particles within a finite range are scattered. So we expect that this result may be a general property for any long-ranged potential scattering problems. This also can be inferred from the fact that in the δ -function potential, the infinitely short-ranged potential, only the s -wave has an information on the scattering such as phase shift. [12]

In conclusion, from the fact that the infinite angular momentum states correspond exactly to the pole in the integral representation of wave function, we find that the incident wave in AB scattering is obtained from the only infinite angular momentum states. It is not certain, however, whether this property holds true for all the long-ranged interaction problems. To ascertain whether this is a general property, some scattering problems other than AB scattering must be investigated. This is now in progress.

REFERENCES

- [1] Y.Aharonov and D.Bohm, *Phys. Rev.* 115, 485 (1959).
- [2] S.N.M.Ruijsenaars, *Ann. Phys.* 146, 1 (1983).
- [3] S.Sakoda and M.Omote, *J. Math. Phys.* 38, 716 (1997).
- [4] C.R.Hagen, *Phys. Rev.* D41, 2015 (1990).
- [5] R.Jackiw, *Ann. Phys.* 201, 83 (1990).
- [6] M.V.Berry, *Eur. J. Phys.* 1, 240 (1980); *Proc. R. Soc. Lond.* A392, 45 (1984).
- [7] T.Takabayashi, *Hadronic Journal Supplement* 1, 219 (1985).
- [8] A.Dasnieres de Veigy and S.Ouvry, *Acad. Sci. Paris. t. 38*, Sèrie II, 19 (1994).
- [9] M.Alvarez, MIT-CTP-2473, hep-th/9510085.
- [10] D.Stelitano, *Phys. Rev.* D51, 5876 (1995).
- [11] I.S.GradshTEyn and I.M.Ryzhik, *Tables and Integrals, Series and Products*. (Academic Press, 1980), p.954.
- [12] R.Jackiw, *M.A.Bég Memorial Volume* ed. A.Ali and P.Hoodbhoy (Singapore, World Scientific).